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Bistability Driven by Correlated Noise: Functional Integral Treatment

J. F. Luciani[1] **and A. D. Verga**[1,2]

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A complete study of non-Markovian effects induced by correlated noise applied to a bistable dynamical system is presented. Starting from the exact functional integral solution of the stochastic equation, it is possible to show that the customary expansion in powers of the characteristic correlation time gives wrong asymptotic results. Other approaches based on a Fokker–Planck equation with a modified diffusion coefficient also fail in reproducing the right long-time behavior of the system. Using a generalized version of instanton calculus of functional integrals, explicit expressions of the invariant measure and transition time between stable fixed points are obtained, in the limit of small noise intensity but arbitrary correlation time. In particular, an original method for extracting the collective degrees of motion has been developed. These analytical results fit, for a large range of parameters, with numerical calculations, giving confidence in the formalism employed.

KEY WORDS: Stochastic equations; non-Markovian effects; functional integral.

1. INTRODUCTION

Stochastic equations of the form

$$\dot{x} = f(x) + y \tag{1}$$

where $f(x)$ generally is a polynomial function and y is a Gaussian noise characterized by a correlation function $\langle y(t)\, y(t')\rangle = \sigma^2 C(t, t')$, are a model for a variety of physical problems. For instance, assume y to be the Ornstein–Uhlenbeck process; then

$$\tau \dot{y} = -y + b(t) \tag{2}$$

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[1] Centre de Physique Théorique, Ecole Polytechnique, 91128 Palaiseau, France.

[2] Member of Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.

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where $b(t)$ is a white noise with intensity σ, and $\tau = 1/\alpha$ is the characteristic correlation time:

$$\langle y(t)\, y(t') \rangle = (\sigma^2/2\tau) \exp(-|t-t'|/\tau)$$

Combining (1) and (2), we obtain

$$\ddot{x} = -(\alpha - f')\, \dot{x} + \alpha f + \alpha b(t) \tag{3}$$

which describes the Brownian motion of a particle in a nonlinear dissipative medium. Note that when α becomes smaller than f', dissipation is not always positive. In the particular case of a force given by $f(x) = \varepsilon x - x^3$, (3) is the normal form, in the presence of noise, of a system undergoing a codimension-2 bifurcation.[3]

Model (1) also applies to the problem of bound states[4] in a random potential in quantum mechanics. Indeed, the Schrödinger equation

$$\psi'' = (y - E)\, \psi \tag{4}$$

for the wave function ψ and negative energy $E < 0$, after a change of variables $\theta = \psi'/\psi$, can be put into the form of a Ricatti equation,

$$\theta' = |E| - \theta^2 + y \tag{5}$$

The drift $|E| - \theta^2$ has a potential function with two fixed points, one stable, the other unstable. The escape from the stable fixed point across the barrier, and then the rotation number, are governed by the random potential. Here, the correlation time is replaced by a coherence length.

In recent years a great deal of interest has been given to bistable dynamical systems perturbed by noise, which, under certain simplifying assumptions, are models of lasers or electronic devices.[1,2,6,22,23] Here, the noise can either be imposed by an external source or be related to environmental perturbations. In both cases it has a correlation time τ, and the basic question is often how the transition time θ between two stable fixed points changes with τ. Because of the lack of rigorous results on the effects of correlated noise on the behavior of bistable systems, a controversy has arisen about the validity of approximate models. In particular, different formulas for the so-called activation rate $S(\tau)$, defined as $\lim_{\sigma \to 0} \sigma^2 \ln \theta$, have been proposed.[5,21,24] The "decoupling ansatz" of Ref. 21 is often used for comparison with numerical results. This ansatz predicts a linear increase of the activation rate with τ. However, as we shall demonstrate, S does increase as $S \sim \tau^2$ for small τ and as $S \sim \tau$ for large τ, but with a slope that markedly differs from the ansatz prediction.

The essential difficulty in treating Eq. (1) is that, for arbitrary

Gaussian processes $\langle y(t)\, y(t') \rangle = \sigma^2 C(t-t')$, in contrast to the Langevin equation, which is related to the Fokker–Planck equation, it is impossible to derive in a closed form a partial differential equation taking exactly into account non-Markovian effects. Therefore, methods currently used are based on the derivation of a master equation,[5] either by systematic expansions in terms of noise intensity or correlation time, or by introducing *ad hoc* closure schemes or ansätze.[6] However, all of these methods fail to describe correctly the long-time dynamics. The point is that the long-time behavior involves nonperturbative quantities, dominated by large-deviation events,[7] while these schemes are at best asymptotic developments valid at short times.

We take a different point of view: we start from the functional integral representation of the transition probability, without trying to derive a master equation.[8] From this exact formal solution, one may apply instanton methods to compute relevant quantities, such as the invariant measure (stationary probability distribution) and the transition time between two fixed points. Rigorous results[7,25] validate the use of this formalism in order to obtain the large-deviation asymptotics. Although in the white-noise case one can readily obtain the activation rates, explicit estimations are much more difficult to achieve in the correlated case.

In Section 2 we derive the master equation and discuss its validity. In Section 3 we briefly review the derivation of the transition probability $P(X, T \mid X_0)$ by means of the integral functional and illustrate how it works in the simple case of a linear stochastic process, which can be calculated exactly. In the general case, only approximate solutions can be obtained, using, for example, the Laplace method to evaluate the relevant paths in the limit of small noise intensity. In this way (Section 4) we compute the so-called activation rate S, which gives the dominant exponential behavior of the stationary probability distribution

$$\mu = \lim_{T \to \infty} P \approx K \exp(-S/\sigma^2) \tag{6}$$

In order to make calculations explicit, we take as a model problem a bistable system, using a piecewise linear function $f(x)$. Next, we are left with the prefactor K calculation. In Section 5, we find the stationary probability function of the final point X. This implies taking into account the Gaussian fluctuations around the extremal ("classical") trajectory, derived from the equation $\delta S = 0$ (Section 5.1). When the final point is a fixed point, a collective degree of motion appears, related to the time translation invariance of the classical trajectory. Gaussian fluctuations diverge, and then it becomes necessary to extract, by means of a gauge condition, the collective coordinate.[9,10] In Section 5.2 we develop a method of collective coordinates adapted to the nonlocal "actions" S encountered in non-Markovian systems. This allows us to compute the stationary probability at the unstable fixed point for a bistable piecewise linear system. In Section 5.3, using the same method, we find the passage time from one stable point to the other. Finally, a series of numerical simulations is performed to compare with analytical results, and an explicit interpolation formula for the transition time, useful in practical comparisons, is given (Section 5.4). Our conclusions are summarized in Section 6.

2. MASTER EQUATION METHOD

A thorough discussion on the derivation of the master equation related to the stochastic process (1) can be found in Refs. 5 and 11. Here we are mainly interested in the limits of validity of this treatment; then deduce the master equation for weak noise intensity, making the underlying assumptions apparent.

Let us start with the Liouville equation satisfied by the distribution function $p_y(x, T)$ before averaging it over noise probability

$$\dot{p}_y + \partial_x(f + y)\, p_y = 0 \tag{7}$$

We want to find an equation for the transition probability $P(x, T)$, defined by the mean value (over noise) $\langle p_y \rangle$. Making the splitting $p_y = P + \rho$, where ρ represents the fluctuating part, we find the system

$$\dot{P} + \partial_x fP = -\partial_x \langle y\rho \rangle \tag{8a}$$

$$\dot{\rho} + \partial_x f\rho + \partial_x[y\rho - \langle y\rho \rangle] = -\partial_x yP \tag{8b}$$

For small noise one may neglect the higher order term $y\rho$ in (8b), obtaining the solution

$$\rho = -[\partial_T + \partial_x f]^{-1}\, \partial_x yP \tag{9}$$

In one dimension the operator $[\partial_T + \partial_x f]^{-1}$ can be inverted explicitly, integrating over the trajectories of the "deterministic" system $dx/dT = f(x)$. Introducing the auxiliary variable $u = \int dx/f$, we can transform (9) into

$$\rho = -\int_0^T \frac{dt'}{f(u)}\, \partial_u P(u - t', T - t')\, y(t') \tag{10}$$

Now, to lowest order in σ^2, but in principle, for arbitrary correlation time, the dominant contribution to ρ comes from the free trajectory

$$P(u - t', T - t') = \frac{f(u)}{f(u - t')}\, P(u, T) \tag{11}$$

Substituting (11) in (10) and then introducing it into (8a), one finally obtains

$$\dot{P} + \partial_x fP = \tfrac{1}{2}\sigma^2 \partial_{xx} D(x) P \tag{12}$$

where

$$D(x) = \int_0^T dt\, C(0, t) \frac{f(x)}{f(x_{-t})} \tag{13}$$

where x_{-t} is the image of x, following the free motion, at time $-t$. In the simplest case of a linear system $f(x) = -\omega x$, the diffusion term reads

$$D(x) = \int_0^T dt\, C(0, t)\, e^{-\omega t} \tag{14}$$

and (12) gives the exact result, as we show below (Section 3). However, in a more general case, for a system having unstable fixed points, the diffusion diverges, and (12) becomes meaningless. Indeed, near the unstable fixed point $x_u = 0$, $f(x) = ax$ $(a > 0)$, and x_{-t} approaches x_u as e^{-at} for arbitrary initial x; therefore, the diffusion behaves as

$$D(x) = \int_0^T dt\, C(0, t)\, e^{at} \tag{15}$$

and this integral is in general divergent. For instance, when the correlation function decays as $e^{-\alpha t}$, we see that (12), which holds formally for arbitrary α, in fact may only describe the behavior of the system for sufficiently small correlation times. In this case, taking, for example, an exponential correlation function, one may expand D in powers of the correlation time[12]

$$D(x) = D_0[1 - \tau f' + O(\tau^2)] \tag{16}$$

For a bistable potential $[f(x) = x - x^3]$, the activation rate becomes

$$\int_0^1 dx \frac{f(x)}{D(x)} = \int_0^1 dx \frac{f(x)}{D_0} (1 - \tau^2 f'^2) \tag{17}$$

and we find that it *decreases* with τ. This contradicts the fact that, at a given unperturbed diffusion σ^2, an increase of the correlation time filters the large fluctuations and therefore would lead to an increase of the activation rate. We demonstrate in Section 4 that effectively the activation rate should grow with τ. Therefore, even for small correlation times the master equation fails to describe correctly non-Markovian effects. The point is that it is not possible to expand a diffusion operator perturbatively in powers of the diffusion coefficient in such a way that the series converge for arbitrary long times (typically $T \gg 1/\sigma$).

3. THE FUNCTIONAL INTEGRAL METHOD

For the sake of completness, we derive in this section the relevant formulas connecting the solution of the stochastic equation (1) to the transition probability $P(X, T | X_0)$ by means of the functional integral formalism.[13-16] We start, as in Section 2, from the noise-dependent probability density $p_y(X, T | X_0) = \delta[X - x_y(T, X_0)]$, where a path between the extremal points X_0 and X is considered; $x_y(T, X_0)$ is a solution of (1). We can represent p_y in terms of a δ-functional in the form

$$p_y(X, T | X_0) = \int Dx\, \delta[x - x_y(t; X_0)] \tag{18}$$

Now, using (1), we can explicitly write the dependence on the noise y,

$$p_y(X, T | X_0) = \int_{X_0}^{X} Dx\, \delta[\dot{x} - f(x) - y]\, J \tag{19}$$

where J is the Jacobian of the transformation $x_y \to y = dx/dt - f(x)$, $x(0) = X_0$,[15,17]

$$J = \det(\partial_t) \exp\left[-\frac{1}{2} \int_0^T dt\, f'(x) \right] \tag{20}$$

Expressing the δ-functional by its Fourier transform, we obtain

$$p_y(X, T | X_0) = \int_{X_0}^{X} Dx\, D\left[\frac{z}{2\pi}\right] \times \exp\left[i \int_0^T dt\, z(\dot{x} - f - y) \right] \exp\left(-\frac{1}{2} \int_0^T dt\, f' \right) \tag{21}$$

where the formal $\det(\partial_t)$ is absent, and consequently (21) is well defined. Next, we have to take the mean value of p_y over the noise probability distribution,

$$P[y] = N \exp\left[-\frac{1}{2\sigma^2} \int_0^T dt\, dt'\, y(t)\, C^{-1}(t, t')\, y(t') \right] \tag{22}$$

C^{-1} is the inverse of C, and $N = [\det(\partial_t / 2\pi\sigma^2)]^{1/2}$ a normalization constant. The resulting Gaussian functional integral is readily performed, giving

$$P(X, T \mid X_0) = \int_{X_0}^{X} Dx\, D\left[\frac{z}{2\pi\sigma^2}\right] e^{-S/\sigma^2} \tag{23}$$

where we have used the scaling $z \to z/\sigma^2$, and

$$S = \frac{1}{2}(z, Cz) - i\int_0^T dt\, z(\dot{x} - f) + \frac{\sigma^2}{2}\int_0^T dt\, \frac{df}{dx} \tag{24}$$

where, in order to simplify the notation, we introduced

$$(z, Cz) = \int_0^T dt \int_0^T dt'\, z(t)\, C(t, t')\, z(t')$$

The fact that the action $S = S[x, z]$ is nonlocal, related to the non-Markovian character of noise y, implies that the stochastic process x cannot be described in a closed form by a Fokker–Planck-like equation. This type of equation can only take into account finite-time effects, or Gaussian fluctuations around stable fixed points. As we have noted in Section 2, the long-time behavior is dominated by large-deviation events, with exponentially small probabilities $\mu \sim \exp(-S/\sigma^2)$, where the global dynamical properties of the system enter into play.

We shall illustrate the functional integral formalism, solving exactly a linear stochastic process for arbitrary correlations and arbitrary time. In this case the transition probability is given by

$$P(X, T \mid X_0) = \int_{X_0}^{X} Dx\, D\left[\frac{z}{2\pi\sigma^2}\right] \exp\left[-\frac{1}{2\sigma^2}(z, Cz) - \frac{i}{\sigma^2}\int_0^T dt\, (\dot{z} - z)\, x\right] \exp\left(\frac{T}{2}\right) \exp[i(ZX - Z_0 X_0)] \tag{25}$$

where we have integrated by parts the term $z(dx/dt)$, $Z = z(T)$, and $Z_0 = z(0)$. The integration over x can be immediatly performed,

$$P(X, T \mid X_0) = \int Dz\, \frac{\det \partial_t}{2\pi\sigma^2}\, \delta[\dot{z} - z] \times \exp\left[-\frac{1}{2\sigma^2}(z, Cz)\right] \exp\left[\frac{T}{2} + \frac{i}{\sigma^2}(ZX - Z_0 X_0)\right] \tag{26}$$

Now, transforming the δ-functional

$$\det(\partial_t)\, \delta[\dot{z} - z] = e^{T/2}\, \delta(z - Z_0 e^t) = e^{-T/2}\, \delta(z - Z e^{t - T}) \tag{27}$$

we reduce the functional integral into an ordinary one

$$P(X, T|X_0) = \int \frac{dZ}{2\pi\sigma^2} \exp\left[\frac{i}{\sigma^2}(X - X_0 e^{-T})\, Z\right] \exp\left[-\frac{D(T)}{2\sigma^2} Z^2\right] \quad (28)$$

where $D(T) = e^{-2T}(e^t, Ce^{t'})$. Doing the Gaussian integration, we finally obtain

$$P(X, TX_0) = \frac{\exp[-(X - X_0 e^{-T})^2/2\sigma^2 D(T)]}{[2\pi\sigma^2 D(T)]^{1/2}} \quad (29)$$

This is an exact result, independent of the explicit form of the correlation function and valid for every time T. The white noise limit $C \to \delta(t - t')$ gives $D(T) = 1 - e^{-2T}$, and for exponential correlation $C = (\alpha/2)\, e^{-\alpha|t|}$ we find

$$D(T) = \frac{1}{2}\frac{1 - e^{-2T}}{1 + \tau} + \frac{\tau}{1 - \tau^2}\,[e^{-(1 + 1/\tau)T} - e^{-2T}] \quad (30)$$

and the stationary probability is given by

$$\mu(X) = \frac{(1 + \tau)^{1/2}}{(\pi\sigma^2)^{1/2}} \exp\left(-\frac{1 + \tau}{\sigma^2} X^2\right) \quad (31)$$

It is worth noticing that for linear stochastic processes we find the same effective diffusion coefficient we had derived for the master equation. This demonstrates that Gaussian fluctuations in the neighborhood of a stable fixed point are appropriately described by this model. However, while the functional integral allows us to derive in a simple way the explicit expression of the transition probability, it is not evident how to solve master equation for arbitrary correlation functions.

4. ACTIVATION RATES FOR BISTABLE SYSTEMS

In what follows we analyze the long-time behavior of a stochastic bistable system. This concerns the study of quantities such as the stationary probability distribution and first passage times between stable fixed points. In the general case, the drift force $f(x)$ is nonlinear, and exact results cannot be obtained. Then, it becomes necessary to develop approximate methods to evaluate functional integrals. However, because of the presence of large deviations it is not possible to solve the stochastic equation using perturbative expansions. In order to extract the leading σ dependence the appropriate method to apply is Laplace evaluation of functional integrals, in the same way as one proceeds in the simpler white noise case.[18,19] That is, the principal contribution to the transition probability, represented by the functional integral (23), comes from the path that makes the action (24) minimal. Therefore, the relevant trajectories satisfy the following Euler–Lagrange equations:

$$\begin{aligned} \dot{x} &= f - i \int_{-T_1}^{T_2} dt'\, C(t, t')\, z(t') \\ \dot{z} &= -f'z, \qquad x(-T_1) = X_0, \qquad x(T_2) = X \end{aligned} \tag{32}$$

Using these equations of motion, one can rewrite the action in terms of the solution $z_c = z_c(t; X_0, X)$,

$$S(X, X_0) = -\tfrac{1}{2}(z_c, Cz_c) \tag{33}$$

We remark that, making the Gaussian integration over z in formula (23), one obtains an action functional $S = \frac{1}{2}((x_c - f_c), C^{-1}(x_c - f_c))$ in the form discussed by Freidlin and Wentzell.[7] However, in our case it is convenient to conserve the variable z, because it greatly simplifies the calculations. In general C^{-1} is not easy to obtain, but the two formulations are obviously equivalent.

To determine $\mu(X)$, one has to consider $T \to \infty$, and then the trajectories pass a long time around fixed points. If X belongs to the attraction basin of a stable fixed point s, it suffices to take into account trajectories that leave s, and then to make $T_1 \to \infty$ in (32). In the particular case that X is itself a stable fixed point ($T_1, T_2 \to \infty$), the relevant trajectory is of the instanton type, but in principle multiinstanton solutions of (32) also contribute to the stationary probability. In general, the way the stationary probability builds up can be modeled by a system of rate equations involving the statistical weights p_i of the stable fixed points s_i. In a system with two stable fixed points separated by an unstable one u, a multiinstanton trajectory is of the form $s_{i1} \to u \to s_{i2} \to \cdots \to s_{iN}$. A Poissonian statistics is established; thus,

$$\dot{p}_1 = -p_1/\theta_1 + p_2/\theta_2, \qquad \dot{p}_2 = -\dot{p}_1 \tag{34}$$

where θ_i are the characteristic transition times. The precise way in which this relaxation dynamics is set up depends on the contribution of each instanton to the stationary probability and will be detailed in Section 5.3.

Knowing the invariant measure of the simple path $s_1 \to X$, $\mu_1(X)$, it is easy to find the stationary probability in the attraction basin of s_1, using (34):

$$\mu(X) = p_1(\infty)\, \mu_1(X) \tag{35}$$

In turn, calculation of the activation rate only implies solving (32) for a simple trajectory connecting the stable to the unstable fixed points, since it is defined by the formula

$$S = \lim_{\sigma \to 0} -\sigma^2 \ln \mu_1(X) \tag{36}$$

and then only the exponential factor in μ contributes.

The integrodifferential equation (32) cannot in general be solved explicitly. Hence, we first study its asymptotic regimes and then we use piecewise linear functions $f(x)$ to solve (32).

Take a bistable system and assume an exponential correlation function $C(t, t')$. We shall find the activation rate for small correlation time. In the neighborhood of the fixed points, $f(x)$ behaves as

$$f \approx -b(x-s), \qquad f \approx a(x-u), \qquad x(-T) \approx s, \qquad x(T) \approx u \qquad (T \to \infty) \tag{37}$$

In the limit $T \to \infty$ the solution of the equation for z is of the form

$$z \approx \begin{cases} e^{-at} & t \to \infty \\ e^{bt} & t \to -\infty \end{cases} \tag{38}$$

This exponential behavior allows us to expand the kernel C in the sense of distributions,

$$C = \delta(t-t') + \tau^2 \ddot{\delta}(t-t') + \cdots \tag{39}$$

It should be emphasized that (39) does not hold in general (if X is not a fixed point). Using (39), the action (33) splits into a white noise part and a perturbation $O(\tau^2)$, $S = S_0 + \tau^2 \delta S$, where

$$\delta S = -\tfrac{1}{2}(z, \tau^2 \ddot{\delta} z) \tag{40}$$

To find the dominant correction in τ^2, it suffices to replace the extremal trajectory $z_0 = -2if$, $dx_0/dt = -f$, related to S_0, in δS. After integration by parts one obtains

$$\delta S = -2\tau^2 \int_s^u dx\, ff'^2 \tag{41}$$

and then, in the small-correlation-time limit, the activation rate is given by

$$S = -2 \int_s^u dx\, f(1 + \tau^2 f'^2) \tag{42}$$

($f < 0$, for $s < x < u$). In the other relevant limit of large correlation time, the kernel behaves as $C \approx 1/(2\tau)$, and the activation rate then reads

$$S = \tau f_M^2 \tag{43}$$

where f_M is the maximum of $|f|$ in the interval (s, u). This obviously corresponds to a static Gaussian drift, with mean value $\sigma^2/2\tau$, which must overcome f_M.

Now we calculate explicitly the activation rate, using as a simple model of a bistable system a piecewise linear function (see Fig. 1)

$$f(x) = \begin{cases} -2(x+1) & -\infty < x \leqslant -5/6 \\ -1/3 & -5/6 < x \leqslant -1/3 \\ x & -1/3 < x \leqslant 0 \end{cases} \tag{44}$$

for $x \leqslant 0$, and $f(x) = -f(-x)$. This function approximates $f(x) = x - x^3$ in such a way that in the limit of white noise one obtains the same result for the activation rate.

The solution of the equation of motion for z, $z = z_c(t)$, is readily obtained, giving

$$z_c(t) = \begin{cases} iZe^{2t} & t < 0 \\ iZ & 0 < t < u \\ iZe^{-(t-u)} & u < t < T \end{cases} \tag{45}$$

where we have introduced three parameters Z, u, and T, which are determined by the matching conditions at the singular points $-1/3$ and $-5/6$, and by the boundary condition $x(T) = X$ ($T \to \infty$, $X \to 0$). Substituting (45)

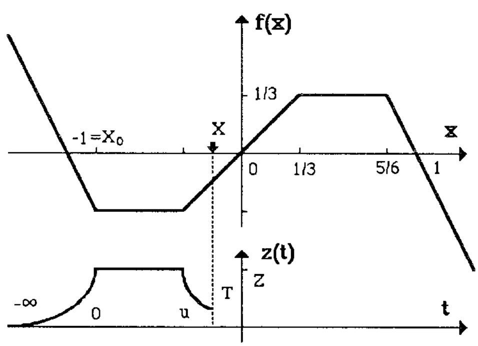

Fig. 1. The piecewise model for a bistable dynamical system used in the analytical computations. In the white noise limit this model gives the same activation rate one would obtain for a drift $f(x) = x - x^3$. The trajectory $z(t)$ of a path $s \to X$ is plotted.

into (8a) and solving the differential equations, one get these conditions in the form of the following transcendental equations:

$$\begin{aligned} Z &= \tau/3I_1 \\ 0 &= (1/6)(I_2/I_1 + 3 - 2u) - 1 \\ X &= (I_3/6I_1 - 1/3)\, e^{T-u} \end{aligned} \tag{46}$$

where

$$\begin{aligned} I_1 &= \frac{\tau}{2}\frac{1+3\tau}{(1+\tau)(1+2\tau)} + \tau^3 \frac{1-e^{-u/\tau}}{(1+\tau)(1+2\tau)} - \tau^2 \frac{e^{-u/\tau}}{(1+\tau)(1+2\tau)}\, e^{-(1+1/\tau)(T-u)} \\ I_2 &= 2u\tau - \tau^3 \frac{(3+4\tau)(1-e^{-u/\tau})}{(1+\tau)(1+2\tau)} - \tau^2 \frac{1-e^{-u/\tau}}{1+\tau}\, e^{-(1+1/\tau)(T-u)} \\ I_3 &= \tau \frac{1+3\tau}{(1+\tau)(1+2\tau)} + 2\tau^3 \frac{1-e^{-u/\tau}}{(1+\tau)(1+2\tau)} - \frac{\tau}{1+\tau}\, e^{-2(T-u)} \\ &\quad + \left(\frac{1+5\tau}{1-\tau} - 2 + 2e^{-u/\tau}\right) \frac{e^{-(1+1/\tau)(T-u)}}{(1+\tau)(1+2\tau)} \end{aligned} \tag{47}$$

In terms of these parameters the action related to the path $-1 \to X \leqslant 0$ reads

$$S(-1, X) = \frac{Z}{2}\left[1 + \frac{u}{3} + Xe^{-(T-u)}\right] \tag{48}$$

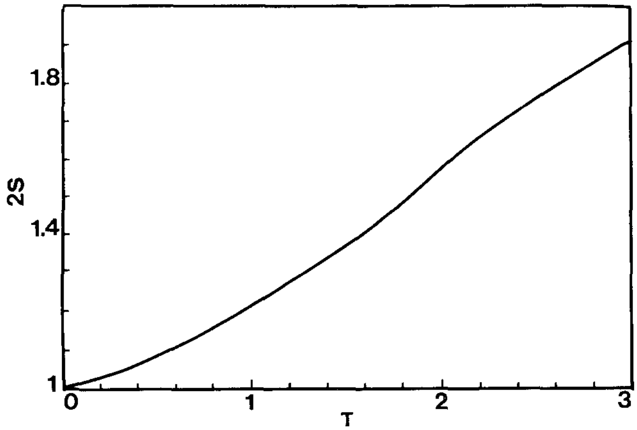

Fig. 2. Pattern of the activation rate $S(\tau)$ for the drift function of Fig. 1.

In particular, the activation rate is $S(\tau) = S(-1, 0)$ (Fig. 2). The asymptotics of $S(\tau)$ are

$$S(\tau) = \begin{cases} \dfrac{1}{2} + \tau^2 \dfrac{3 + 4\tau}{3(1+2\tau)(1+\tau)} & \tau \ll 1 \\ \dfrac{1}{2} + \dfrac{\tau}{9} & \tau \gg 1 \end{cases} \tag{49}$$

This behavior is consistent with (42) and (43).

5. STATIONARY PROBABILITY AND TRANSITION TIME

5.1. The Invariant Measure outside Unstable Fixed Points

Now we turn our attention to the computation of the invariant measure $\mu(X)$ if X is not a fixed point. We take $-1/3 \leqslant X < 0$. The leading exponential contribution has been computed in Section 4; so we are left to study the Gaussian fluctuations around the classical trajectory. Let

$$x = x_c + \xi, \qquad z = z_c + \eta \tag{50}$$

where (x_c, z_c) is a solution of the equations of motion in the interval $(-\infty, T)$. The invariant measure at X is then given by

$$\mu(X) = P(X, T \mid -\infty) = K \exp(-S_c/\sigma^2) \tag{51}$$

At leading order in σ, the prefactor K reads

$$K = \left[\exp\left(-\frac{1}{2}\int dt\, f_c'\right)\right] \int D\left[\frac{\eta}{2\pi\sigma^2}\right] D\xi \exp\left[-\frac{1}{2\sigma^2}(\eta, C\eta)\right] \\ \times \exp\left\{\frac{i}{\sigma^2}\int dt\, \eta[\dot{\xi} - f_c'\xi] - \frac{z_c f_c''}{2}\xi^2\right\} \tag{52}$$

where the boundary conditions are $\xi(-\infty) = \xi(T) = 0$. We remark that f_c'' can be written

$$f_c'' = [2/|\dot{x}_c(0)|]\,\delta(t) + [1/|\dot{x}_c(u)|]\,\delta(t-u) \tag{53}$$

for our piecewise linear model, and then the conjugate variable η must be discontinuous at $t = 0, u$. This allows us readily to transform the functional integral (52) into an ordinary one, integrating first over ξ and then over η.

A detailed account of the calculation is given in Appendix A. The integration over ξ generates a δ-functional of the form

$$\prod_{t\in(-\infty,0)} \delta[\dot{\eta}+f_c'\eta] \prod_{t\in(0,u)} \delta[\dot{\eta}+f_c'\eta] \prod_{t\in(u,T)} \delta[\dot{\eta}+f_c'\eta] \tag{54}$$

One can pass from $d\eta/dt$ to η by solving the differential equation for η. Introducing appropriate jumps Δ_1 and Δ_2 at the singular points 0 and u (cf. Appendix A), one has

$$\eta_\Delta(t)=\begin{cases} he^{2t} & t\in(-\infty,0] \\ h+\Delta_1 & t\in(0,u] \\ (h+\Delta_1+\Delta_2)\,e^{-t} & t\in(u,T] \end{cases} \tag{55}$$

where h is an intermediate value, $\eta(0)=h$. The Jacobian of this transformation is given by

$$\exp\left(\frac{1}{2}\int_{-\infty}^{0} dt\, f_c'\right)\exp\left(-\frac{1}{2}\int_{0}^{T} dt\, f_c'\right) \tag{56}$$

where care should be taken with the sense of time propagation: forward in the interval $(-\infty, 0)$, backward in $[0, T]$. After these steps, we finally find

$$K=\int dh\,\frac{d^2\Delta\, d^2\xi}{(2\pi\sigma^2)^3}\exp[-(T-u)]\exp\left[-\frac{B(h,\Delta,\xi)}{\sigma^2}\right] \tag{57}$$

where B is the bilinear form in h, Δ, and ξ:

$$B=1/2(\eta_\Delta, C\eta_\Delta)-i\Delta_1\xi_1-i\Delta_2\xi_2-3Z\xi_1^2-3/2Z\xi_2^2 \tag{58}$$

5.2. Collective Coordinates and the Invariant Measure at the Unstable State

In the limit $T\to\infty$, the final point X approaches the unstable fixed point and the bilinear form B becomes degenerate, making the Gaussian integral (57) meaningless. The appearance of an exponentially small eigenvalue is related to the approximate time translation invariance (small σ) of the classical trajectory. Thus, in the vectorial space displayed by B, a direction arises where the assumption of small Gaussian fluctuations around the classical trajectory breaks down. This direction is defined by the eigenvector $\mathbf{x}_0=(dx_c/dT^*, dz_c/dT^*)$, where T^* parametrizes equivalent time-translated trajectories. It then becomes necessary to treat separately the degree of freedom associated with time translations.

Following the method of collective coordinates,[9,10] we introduce a "gauge" condition of the form $\int dT^* \, \delta[\langle \mathbf{x}_1 | \mathbf{x}_0 \rangle] \langle \mathbf{x}_0 | \mathbf{x}_0 \rangle \approx 1$, where $\langle \cdot | \cdot \rangle$ is an arbitrary scalar product and $\mathbf{x}_1$ is the vector (ξ, η). It is worth noticing that, due to the fact that the action is nonlocal, the relation between $\langle \mathbf{x}_0 | \mathbf{x}_0 \rangle$ and S is not trivial. Moreover, in contrast to the commonly encountered situation where a global factor T^* is obtained, the presence of the Jacobian term $\sim e^{-T}$ introduces an explicit dependence on T^* (after the shift $T \to T - T^*$). This means that the translational invariance is broken at σ^2 order. Therefore, the limit $T \to \infty$ must be carefully carried out, taking into account the action at finite time.

The simplest way to proceed is to consider the infinite-time classical trajectory and to truncate it at $t = T^* - T$, T^*, adding jumps to match with boundary conditions. Taking T fixed and varying T^*, one obtains trajectories that differ in their discontinuities at the boundaries. The action S^* related to this truncated motion is $S^* = 2S_\infty + \frac{1}{2}(z_c, Cz_c)_{\mathbf{I} \times \mathbf{I}}$, where S_∞ is the action for infinite time and the second term is the action in the interval $\mathbf{I} = (T^* - T, T^*)$ (see Appendix B for a complete calculation). In this way, fluctuations can be treated as before, using vanishing boundary conditions, without adding supplementary contributions coming from integration by parts terms. Putting all this together, the limit $T \to \infty$ of Eq. (57) is written in the following form:

$$\mu(0) = \lim_{T \to \infty} \int_0^T dT^* \exp(-T^*) \exp\left[-\frac{S^*(T^*)}{\sigma^2}\right] \times \int_{-\infty}^{\infty} dh \frac{d^2\Delta \, d^2\xi}{(2\pi\sigma^2)^3} \delta[\langle \mathbf{x}_1 | \mathbf{x}_0 \rangle] \langle \mathbf{x}_0 | \mathbf{x}_0 \rangle \exp\left(-\frac{B}{\sigma^2}\right) \tag{59}$$

where the action S^* is given by

$$S^* = S_\infty + 2Z^2 \left\{ \frac{\alpha}{\alpha - 1} \frac{v^2}{2} - \left[\frac{3\alpha}{\alpha - 1} - 2(1 - e^{-\alpha u}) \right] \frac{v^{\alpha+1}}{(\alpha+1)(\alpha+2)} \right\} \tag{59a}$$

v is $\exp(-T^*)$ and B is the bilinear form of Eq. (58) in the limit $T \to \infty$; we write it after formal (analytic continuation is necessary) integration over ξ_n,

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{pmatrix}$$

$$B_{11} = \sum_{n=1}^{6} c_n, \qquad B_{12} = c_3 + c_5 + c_6 + \tfrac{1}{2}(c_2 + c_4), \qquad B_{13} = c_6 + \tfrac{1}{2}(c_4 + c_5)$$

$$B_{22} = c_3 + c_5 + c_6 + d_1, \qquad B_{23} = c_6 + \frac{c_5}{2}, \qquad B_{33} = c_6 + d_2 \tag{59b}$$

where

$$
\begin{aligned}
&c_1 = \frac{\alpha}{4(\alpha+2)}, \qquad c_2 = \frac{1-e^{\alpha\mu}}{\alpha+2}, \qquad c_3 = \frac{u}{2} - \frac{1-e^{-\alpha u}}{\alpha} \\
&c_4 = \frac{\alpha e^{-\alpha u}}{(\alpha+1)(\alpha+2)}, \qquad c_5 = \frac{1-e^{-\alpha u}}{\alpha+1}, \qquad c_6 = \frac{\alpha}{2(\alpha+1)} \\
&d_1 = -\frac{1}{4}\left(\frac{\alpha}{\alpha+2} + 1 - e^{-\alpha u} + \frac{\alpha e^{-\alpha u}}{\alpha+1} - \frac{2}{3Z}\right) \\
&d_2 = -\frac{1}{2}\left(\frac{\alpha}{\alpha+1} + 1 - e^{-\alpha u} + \frac{\alpha e^{-\alpha u}}{\alpha+2} - \frac{2}{3Z}\right)
\end{aligned}
\tag{59c}
$$

We recall the origin of each term in formula (59): there is a contribution of the classical trajectory to $\mu(0)$ given by S^*, which takes into account the finite-time effects; fluctuations around the classical trajectory are given by the second-order action B, which, after reduction of the original functional integral into an ordinary one integrating the linear pieces of f, becomes a bilinear form depending on the coordinates associated with the discontinuity points of the perturbed (Gaussian) trajectory; the collective motion is separated from the other modes by means of the δ-function, which allows us to treat exactly the approximate time translation invariance of the classical trajectory in such a way as to obtain a finite result; the integration over the collective mode is given by $dT^* \exp(-T^*)$, where $\exp(-T^*)$ is the Jacobian factor coming from the $D\eta$ functional integration.

Equation (59) straightforwardly reduces to an ordinary integral representation of $\mu(0)$; the Gaussian part gives $|\det B_R|^{-1/2}$, where B_R is the bilinear form reduced to the subspace "orthogonal" to the zero mode. Taking $h=0$ as the gauge condition, one obtains

$$
\begin{aligned}
&\mu(0) = \frac{2^{2/3} Z}{(\pi\sigma^2 d_1 d_2)^{1/2}} |\det B_R|^{-1/2} \int_0^1 dv \exp -\frac{S^*(v)}{\sigma^2} \\
&B_R = \begin{pmatrix} B_{22} & B_{23} \\ B_{23} & B_{33} \end{pmatrix}
\end{aligned}
\tag{60}
$$

When the correlation time crosses the value 1, the bistable dynamical system passes from a state of normal dissipation to another state of negative dissipation. The invariant measure at the stable fixed point is insensitive to this change, and fluctuations remain essentially Gaussian. In contrast, the stationary probability at the unstable fixed point behaves differently, depending on the actual dissipation regime. While for short correlation time, $\mu(0)$ depends on the noise intensity in the same way as

$\mu(-1)$, for large correlation times, non-Gaussian fluctuations dominate the prefactor, and an anomalous σ dependence of the form $\sigma^{(1-\tau)/(1+\tau)}$ appears. This can be seen using formula (59a): for $\alpha \ll 1$ the action is dominated by the $v^{\alpha+1}$ term, which after integration gives

$$\mu(0)/\mu(-1) \sim C_1(\alpha)\, \sigma^{(1-\tau)/(1+\tau)} \exp(-S_\infty/\sigma^2)$$

where $C_1(\alpha)$ is a constant, which only depends on the form of f and on the correlation time. In contrast, when $\alpha \gg 1$ the term v^2 dominates and one has

$$\mu(0)/\mu(-1) \sim C_2(\alpha) \exp(-S_\infty/\sigma^2)$$

with normal prefactor.

5.3. First Passage Time

For a white noise, the transition time between the two stable fixed points is generally computed as twice the exit time from the domain $(-\infty, 0]$, using the Kolmogorov backward equation. But in our case the simplest way is to compute the first instanton contribution and study the building up of the dynamics of the transition probability $P(T) = P(s_2, T/s_1)$. We shall demonstrate that a Poissonian statistics is established, which means that the transition probability as a function of T is exponentially distributed. Therefore, it satisfies a differential equation of the type of (34). Assuming a Poissonian statics, it suffices to expand $P(T)$ in powers of T, and retaining the first term, one obtains the transition time from the formula

$$P(s_1, T \mid s_2) \approx \frac{T}{\theta}\, \mu(s_2) \tag{61}$$

where $\mu(s_2)$ is the measure corresponding to local Gaussian fluctuations around s_2. At variance to what happened with $\mu(0)$, here two collective degrees of motion should be taken into account. The relevant trajectory is essentially formed by the instanton $s_1 \to u$ followed by the deterministic $dx/dt = f$ motion connecting u to s_2. In order to treat the limit $T \to \infty$ properly, let us consider the truncated trajectory of Fig. 3. The labeled minus $(-)$ trajectory is the instanton, the plus $(+)$ trajectory is the deterministic motion. The two collective modes are parametrized by T_1 and T_2. We denote by b_1 and b_2 the slopes of $f(x)$ in the regions T_1 and T_2, respectively. The position T_0 of the central discontinuity is irrelevant. An outline of the computation together with the explicit white noise case is given in Appendix C. Let us summarize the principal steps:

1. The collective mode corresponding to a global time shift (T_2) contributes, as usual, with a factor T.

2. As for the computation of $\mu(0)$ in Section 5.2, the invariance related to a dilatation of T_1 is broken by the Jacobian term. However, in contrast to what hapened with $\mu(0)$, this does not lead to an anomalous scaling of the prefactor. This mode contributes through an integral

$$\int \frac{dT_1}{\sigma^2} \exp(-b_1 T_1) \exp\left[-\frac{C}{\sigma^2} \exp(-b_1 T_1)\right] \tag{62}$$

where C is a constant.

3. One can factorize the $(+)$ and $(-)$ contributions with an appropriate choice of the gauge conditions. The $(+)$ contribution builds up $\mu(s_2)$. The $(-)$ sector contributes through the quadratic form B of Eq. (58) with the same gauge condition as in (59).

In this way we get the simple result

$$P(s_2, T \mid s_1) = (T/2\pi)\, \mu(s_2)(\det B_R)^{-1/2} \exp(-S_\infty/\sigma^2) \tag{63}$$

Thus

$$\theta = 2\pi(\det B_R)^{1/2} \exp(S_\infty/\sigma^2) \tag{64}$$

Now let us return to the multiinstanton contributions. They arise from all trajectories connecting s_1 to s_2, crossing the unstable fixed point, or "reflecting" on it. In this last case, a factor -1 appears, imposed by the conservation of probabilities, and should be obtained by a suitable analytic continuation, which we do not discuss here. In this way, one can check that the formal expansion of the solutions of (34) in powers of T is obtained. In

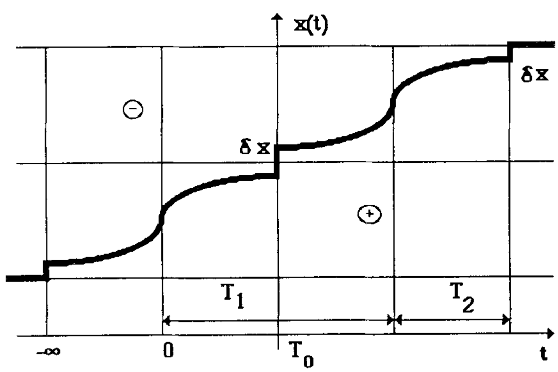

Fig. 3. A scheme of the truncated trajectory used in the computation of the transition time. The two parameters T_1 and T_2 related to the collective degrees of motion are specified.

our case θ_1 and θ_2 are equal, but the general asymmetric potential can be treated as well. In such a case θ_1 and θ_2 would be given by a formula like (64), with B and S replaced by similar quantities corresponding to each potential well. For instance, consider the dynamics up to second order in T. We obtain for the statistical weights P_1 and P_2 (starting from s_1)

$$\begin{aligned}
P_1 = 1 & \qquad (s_1 \to s_1) \\
-T/\theta_1 & \qquad (s_1 \to u \to s_1) \\
+T^2/2\theta_1^2 & \qquad (s_1 \to u \to s_1 \to u \to s_1) \\
+T^2/2\theta_1\theta_2 & \qquad (s_1 \to u \to s_2 \to u \to s_1) \\
P_2 = T/\theta_1 & \qquad (s_1 \to u \to s_2) \\
-T^2/2\theta_1^2 & \qquad (s_1 \to u \to s_1 \to u \to s_2) \\
-T^2/2\theta_1\theta_2 & \qquad (s_1 \to u \to s_2 \to u \to s_2)
\end{aligned} \tag{65}$$

where we have written in parentheses the trajectories related to each contribution. We see that (65) effectively reproduce the first terms of the series solution (in powers of T) of Eqs. (34).

5.4. Numerical Simulations

We performed a series of numerical simulations of Eqs. (1)–(2) using a second-order Runge-Kutta method.[20] We made diagnostics of the invariant measure and of the transition time. Typically 10^8 time iterations are necessary to reach statistically relevant results (Poissonian statistics on the transition time).

Figure 4 shows the stationary probability at the unstable point. We plotted $\Phi = 2\sigma^2 \ln[\mu(s)/\mu(u)]$ as a function of $2\sigma^2$ for several values of τ. The region of small noise and large correlation time remained unexplored due to the excessive computational time required. Our purpose, rather than making a thorough numerical analysis of the problem, is to test the theory. We proved the performance of the analytical calculation in the region where it is more sensitive to the details of its dependence on the correlation time, that is, the region around $\tau \approx 1$, which distinguishes the negative from the positive dissipation regime. Moreover, the contribution of nonlinear terms to $\mu(u)$ is more important than outside the unstable fixed point, where the invariant measure is dominated by local trajectories. We also compared the calculated $\mu(x)$ $(x \neq u)$ with simulations, obtaining complete agreement.

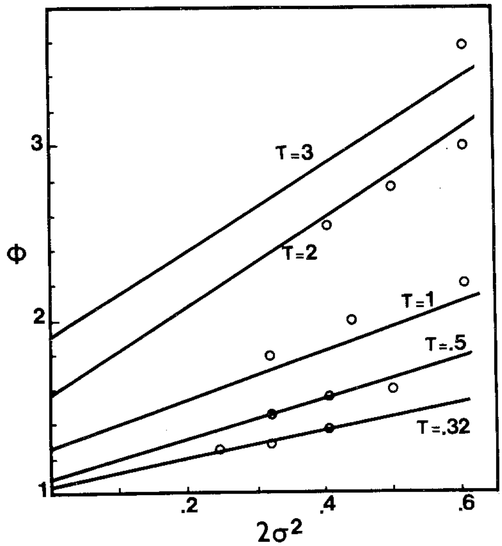

Fig. 4. Numerical simulation of Eqs. (1) and (2) with $f(x) = x - x^3$. Invariant measure at the unstable fixed point, $\Phi = 2\sigma^2 \ln[\mu(s)/\mu(u)]$, as a function of noise intensity σ and correlation time τ. Straigth lines are the analytical results, open circles are the numerical values.

In general the theory agrees very well with numerical calculations, especially for relatively small noise, for which discrepancies are at most on the order of 10%. With regard to transition times, plotted in Fig. 5 $[\Theta = 2\sigma^2 \ln(\theta/\theta_0)$, where θ_0 is the white noise transition time], the agreement is also acceptable. The invariant measure seems to be more sensitive to the adopted form of $f(x)$ than the transition time. Numerical

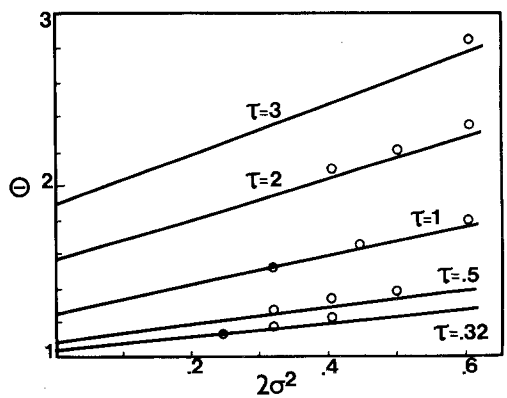

Fig. 5. The same as in Fig. 4 for the transition time, $\Theta = 2\sigma^2 \ln(\theta/\theta_0)$.

simulations using the same piecewise linear function as in the theory match exactly with analytical computations for both invariant measure and transition time.

We give an interpolation formula for the transition time, which may be useful for comparisons with numerical computations of a bistable system $dx/dt = f(x)$:

$$\theta \approx 2\pi \left[\frac{1 + [|f'(s)| + |f'(u)|]\, \tau}{|f'(s)\, f'(u)|} \right]^{1/2} \exp\left[\frac{S_0}{\sigma^2} \frac{(1 + a\tau)(1 + b\tau)}{1 + (a + b)\, \tau} \right] \tag{66}$$

where

$$a + b = \frac{2}{f_M^2} \int_s^u dx\, ff'^2, \qquad ab = \frac{1}{S_0} f_M^2$$

and $f'(s)$ and $f'(u)$ are the derivatives of f at the stable s point and at the unstable u point, respectively, and $S_0 = 2 \int_s^u dx\, f$ is the activation rate for the white noise. This estimation is obtained from the exact expression of the transition time given in Ref. 8, where f is approximated by a piecewise linear function with only three pieces. We have kept the same functional form for $\theta(\tau)$. For the activation rate, the constants are chosen in order to match with the asymptotic values (42) and (43). The prefactor is indeed dominated by the local values $f(u)$ and $f(s)$.

In recent work Hanggi *et al.*[6,21] have treated the same stochastic problem, using a closure hypothesis to obtain a Markovian approximate Fokker–Planck equation. They find that non-Markovian effects can be described by a modification of the diffusion coefficient

$$D(\tau) = D(0)/(1 - \tau\langle f'\rangle) \tag{67}$$

and therefore they conclude that the magnitude of the ratio of the stationary probability taken at the stable state and the unstable state or the transition time is essentially dominated by the activation rate. Although their ansatz produces activation rates that grow with correlation time, improving the usual theory based on an expansion around the white noise limit, it is quantitatively unsatisfactory. First, the asymptotic form (43) for large τ is obvious in the static limit, and is not reproduced by this ansatz. Second, their activation rate, being linear in τ, fails to appropriately describe the right $S = S_0[1 + O(\tau^2)]$ behavior for small correlation time clearly seen in Figs. 4 and 5. In fact, the contribution of the prefactor is also quantitatively important for describing the system's behavior. For instance, the invariant measure at the unstable state develops an anomalous σ dependence, which comes from the prefactor and is absent in the transition time. On the other hand, the closure hypothesis simply predicts a constant prefactor for the transition time, thus giving parallel straight lines in a figure such as Fig. 5, in contradiction with simulations. Therefore, we may conclude that the overall pattern of the invariant measure and transition time as functions of noise intensity and correlation time is influenced by the prefactor for the numerically accessible range of parameters. Only in the noiseless limit does one recover the dependence on the activation rate, behavior that obviously cannot be isolated in numerical calculations.

6. CONCLUSIONS

The long-standing problem of a bistable dynamical system undergoing non-Markovian random perturbations has been studied using the formalism of the functional integral. We have demonstrated that starting from the exact transition probability allows us to overcome the inherent limitations of other methods such as those using master equations or the closure hypothesis. The functional integral directly gives the dominant dependence on noise intensity for arbitrary correlation time, thus taking properly into account the large-deviation contributions, which determine the long-time behavior of the system.

We analyzed the invariant measure and the transition time, first deriving the so-called activation rate (exponential asymptotic) and then the fluctuations around relevant trajectories (prefactor contribution). At the level of the activation rate the shortcomings of the master equation were made apparent, demonstrating that this approach does not succed in describing the system's behavior near unstable fixed points. For instance, the first contribution of non-Markovian effects, assuming small correlation times, derived from the master equation, gives a correction to the white noise result with the bad sign. On the other hand, the closure scheme, giving an activation rate increasing with correlation time, fails to reproduce the right $O(\tau^2)$ asymptotic.

The computation of the prefactor contribution comes up against the problem of the collective degrees of motion resulting from the time translation invariance of the classical trajectory. In order to describe the fluctuations appropriately it becomes necessary to introduce a constraint (gauge condition) which selects among all the equivalent trajectories such that the deviation between the actual trajectory and the approximate classical one is minimized. In contrast to the usual quantum mechanical situation, in the classical problem, where a stationary probability distribution exists, two types of collective motion can be distinguished (in relation to a bistable potential). While the motion connecting the stable fixed points generates a global factor T in the same way as it appears in quantum systems, the motion from a stable fixed point to the unstable one gives a finite contribution. In the case of the invariant measure at the unstable fixed point a qualitative change of behavior arises-when the value $\tau = 1$ is crossed, which (linearly) separates the negative and positive dissipation regimes. For $\tau > 1$ there is an anomalous dependence on the noise intensity of the form $\sigma^{(1-\tau)/(1+\tau)}$, which means that the usual perturbative series in powers of σ do not work.

A series of numerical computations was performed and compared with analytical results. A satisfactory overall agreement between simulations and the functional integral theory was obtained. Is is worth noticing that the role of the prefactor is important to an understanding of the behavior of the non-Markovian system. For instance, the anomalous form of the invariant measure at the unstable point seems difficult to obtain by any closure scheme.

APPENDIX A

Consider the Gaussian functional integral for the prefactor, which we rewrite here for the sake of completeness:

$$K = \exp\left(-\frac{1}{2}\int dt\, f_c'\right) \int D\left[\frac{\eta}{2\pi\sigma^2}\right] \exp\left[-\frac{1}{2\sigma^2}(\eta, C\eta)\right] I(\eta) \quad \text{(A1)}$$

where

$$I(\eta) = \int_0^0 D\xi \exp\left\{\frac{i}{\sigma^2}\int dt\, \eta[\dot{\xi} - f_c'\xi] - \frac{z_c f_c''}{2}\xi^2\right\} \quad \text{(A2)}$$

and closed trajectories $\xi(-\infty) = \xi(T) = 0$ are taken. For a piecewise linear function with N singular points located at $x_n = x(t_n)$ $(n = 1, 2,..., N)$, characterized by the set $(b_0,..., b_N)$ of slopes between x_n and x_{n-1}, we have

$$i\int dt\, z_c f_c''\xi^2 = -\sum_{n=1}^{N} a_n \xi_n^2 \quad \text{(A3)}$$

where

$$a_n = \frac{Z_n}{|\dot{x}_n|}(\Delta f')_n \quad \text{(A4)}$$

with $(\Delta f')_n = b_n - b_{n-1}$ the discontinuity jump of f' at x_n and

$Z_n = -iz_c(t_n)$. Hence, the Hamilton equations related to this second-order action are

$$\begin{aligned} \dot{\xi} - f_c' \xi &= -i \int dt' \, C(t, t') \, \eta(t') \\ \dot{\eta} + f_c' \eta &= -\sum_{n=1}^{N} a_n \, \delta(t - t_n) \, \xi \end{aligned} \tag{A5}$$

From the second of these equations we note that conjugated to each variable ξ_n a discontinuity appears in the function $\eta(t)$, which we denote Δ_n. In contrast with the usual method for integrating Gaussians, via integration of the motion equations (A5), we shall use a simpler procedure, taking advantage of the special form of f, consisting in integrating first over ξ and then over η. In order to make the ξ integration we start by integrating by parts the term $(d\xi/dt)\,\eta$, which gives boundary terms of the form $\xi_n \Delta_n$. We get

$$I(\eta) = \int_0^0 D\xi \exp\left\{ -\frac{i}{\sigma^2} \int dt \, [\dot{\eta} + f_c' \eta] \, \xi \right\} \exp\left[\frac{1}{\sigma^2} \sum_{n=1}^{N} \left(i\xi_n \Delta_n + \frac{a_n}{2} \xi_n^2 \right) \right] \tag{A6}$$

and hence

$$I(\eta) = C\delta_{(N)}(\dot{\eta} + f_c' \eta) \int d^N \xi \exp\left[\frac{1}{\sigma^2} \sum_{n=1}^{N} \left(i\xi_n \Delta_n + \frac{a_n}{2} \xi_n^2 \right) \right] \tag{A7}$$

Here C is a (singular) constant of the form $[\det_{(N)}(2\pi\sigma^2 \partial_t)]$, where the index N means that the factors at times t_n are excluded. We note that, due to the fact that the signs of the a_n are arbitrary, it is better to leave the $d^N\xi$ integration implicit. Now we transform the δ-functional, expressing it as a function of $\eta_\Delta(t)$, the solution of (A5) between discontinuities:

$$\eta_\Delta(t) = \begin{cases} h e^{b_0 t} & t \in (-\infty, t_1) \\ (h + \Delta_1) \, e^{b_1 (t - t_1)} & t \in (t_1, t_2) \\ \cdots & \\ (h + \Delta_1 + \Delta_2 + \cdots + \Delta_N) \, e^{b_N (t - t_N)} & t \in (t_N, T) \end{cases} \tag{A8}$$

This transformation causes a Jacobian term to appear. For forward time propagation $\partial_{t'}^{-1} = \theta(t - t')$, we obtain

$$J^{-1} = \det{}_{(N)}^{-1}(\partial_t) \exp\left(\frac{1}{2} \int dt \, f' \right)$$

whereas for backward propagation one has $\partial_{t'}^{-1} = -\theta(t'-t)$, and the sign of the exponent changes. We used the identity

$$\det(1+M) = \exp[\operatorname{tr} \ln(1+M)]$$

and $\theta(0) = 1/2$ (Stratonovich integral). Having taken the "initial" condition at the intermediate time $t = t_1$, we find the Jacobian

$$J^{-1} = \det_{(N)}^{-1}(\partial_t) \exp\left(\frac{1}{2}\int_{-\infty}^{t_1} dt\, f' - \frac{1}{2}\int_{t_1}^{T} dt\, f'\right) \tag{A9}$$

and $I(\eta)$ finally becomes

$$I(\eta) = \prod_{(N)} (2\pi\sigma^2) \exp\left(\frac{1}{2}\int_{-\infty}^{t_1} dt\, f' - \frac{1}{2}\int_{t_1}^{T} dt\, f'\right) \delta_{(N)}(\eta - \eta_\Delta)$$
$$\times \int d^N\xi \exp\left[\frac{1}{\sigma^2}\sum_{n=1}^{N}\left(i\xi_n \Delta_n + \frac{a_n}{2}\xi_n^2\right)\right] \tag{A10}$$

The remaining integration over η outside the singular points is now trivial due to the presence of the δ in (A10). Inserting (A10) in (A1), we obtain

$$K = \frac{\exp(-\int_{t_1}^{T} dt\, f')}{(2\pi\sigma^2)^{N+1}} \int_{-\infty}^{\infty} dh \prod_{n=1}^{N} d\xi_n\, d\Delta_n \exp\left(-\frac{B}{\sigma^2}\right) \tag{A11}$$

where $B = B(\xi, \Delta)$ is the bilinear form

$$B(\xi, \Delta) = \frac{1}{2}(\eta_\Delta, C\eta_\Delta) - \sum_{n=1}^{N}\left(i\Delta_n \xi_n + \frac{a_n}{2}\xi_n^2\right) \tag{A12}$$

which at finite time T is positive-definite.

APPENDIX B

In the limit $T \to \infty$ the classical trajectory (x_c, z_c) becomes invariant against time translations. The action remains constant when trajectories are displaced, $(x_c, z_c) \to (x_c^*, z_c^*)$ for $T \to T - T^*$, with T^* the time translation parameter and $x_c^* = x_c(t, T^*)$. The assumption of Gaussian fluctuations around this trajectory is then meaningless, due to the presence of a vanishing eigenvalue. To overcome the difficulties related to this zero eigenvalue, it is convenient to introduce a "gauge" condition which makes it possible to treat separately time invariance and the other (Gaussian) degrees of motion. In fact, for the perturbation expansion to be well defined, it is necessary to choose T^* so that the shifted trajectory (x_c^*, z_c^*) most nearly fits the actual trajectory (x, z). Thus, we define a functional Φ, which measures the distance between these trajectories,

$$\Phi(x, z; T^*) = \int dT^* \, Q(|z - z_c^*|) + \sum_{n=1}^{N} \alpha_n [x_c(t_n^*) - x_n]^2 \tag{B1}$$

where Q is an arbitrary quadratic function of its argument and the α_n are arbitrary constants. For convenience we have chosen to approach x by $x_c^*(t)$ at the shifted discontinuity point $t = t_n^*$. The minimum of this functional gives the appropriate value of T^*. To lowest order, the condition of the minimum of Φ can be written in the form of the constraint

$$\begin{aligned} &\delta(\langle \mathbf{x}_1 | \mathbf{x}_0 \rangle) \langle \mathbf{x}_0 | \mathbf{x}_0 \rangle \\ &\quad = \delta \left[\langle \dot{z}_c^* | z - z_c^* \rangle_Q + \sum_{n=1}^{N} \alpha_n | x_c(t_n^*) - x_n | \, \dot{x}_c(t_n^*) \right] \\ &\quad \times \left[\langle \dot{z}_c^* | \dot{z}_c^* \rangle + \sum_{n=1}^{N} \alpha_n \, |\dot{x}_c(t_n^*)|^2 \right] \end{aligned} \tag{B2}$$

where we used $\partial_{T^*} t_n^* = 1$, and $\langle \cdot | \cdot \rangle_Q$ is the scalar product associated with the metric Q.

Now we introduce into the functional integral for the invariant measure the shift $x = x_c^* + \xi$, $z = z_c^* + \eta$; thus, we must calculate the action $S^* = S(x_c^*, z_c^*)$ on a finite interval, making explicit its dependence on T^*. We use a truncated version of the infinite time trajectory, centered at the origin, on the interval defined by $\mathbf{I} = (T^* - T, T^*)$, adding jumps δx at the edges in such a way as to match with the boundaries. Specifically, we have

$$S^* = \tfrac{1}{2}(z_c, C z_c)_{\mathbf{I} \times \mathbf{I}} - (z_c, C z_c)_{\mathbf{I} \times \mathbf{R}} - i[z \, \delta x] \tag{B3}$$

The boundary term

$$[z \, \delta x] = z \, \delta x |_{T^*} - z \, \delta x |_{T^* - T}$$

is calculated using the asymptotic form of $f(x)$ near fixed points, and the result is equal to $-i(z_c, C z_c)_{\mathbf{C} \times \mathbf{I}}$ ($\mathbf{C}$ is the complement of $\mathbf{I}$ in the axis $\mathbf{R}$). We obtain for S^* the expression

$$S^* = 2 S_\infty + \tfrac{1}{2}(z_c, C z_c)_{\mathbf{I} \times \mathbf{I}} \tag{B4}$$

Therefore, using the constraint (B4) and taking into account only the lowest order in ξ and η, we obtain the invariant measure at the unstable fixed point in the form

$$\mu(0) = \lim_{T \to \infty} \int_0^T dT^* \left[\exp\left(-\int_{\mathbf{I}} dt\, f_c' \right) \right] \int D\xi\, D\left[\frac{\eta}{2\pi\sigma^2} \right] \times \delta(\langle \mathbf{x}_1 | \mathbf{x}_0 \rangle) \langle \mathbf{x}_0 | \mathbf{x}_0 \rangle \exp - \frac{(S^* + L + B[\xi^*, \eta^*])}{\sigma^2} \tag{B5}$$

where $B^* = B[\xi^*, \eta^*]$ is a bilinear form for the shifted fluctuations

$$B[\xi, \eta] = \frac{1}{2}(\eta, C\eta) - i \int dt \left[\eta(\dot{\xi} - f_c'\xi) - \frac{z_c f_c''}{2} \xi^2 \right] \tag{B6}$$

It can be easily demonstrated that the linear term L

$$L = -i \int dt\, \eta(\dot{x}_c - f) = -i(\eta, C z_c)_{\mathbf{I} \times \mathbf{R}}$$

which arises from the fact that the trajectory we use does not satisfy the equation of motion, is negligible at the limit of interest. It should be noted, however, that in the case of a trajectory which joins the two stable fixed points, this term contributes to the transition probability and must then be retained (cf. Section 5.3).

We note that the choice of the constraint on x using just the discontinuity points allows us to proceed as before, in the sense that integration over ξ can be made directly. After functional integration (B5) reads

$$\mu(0) = \lim_{T \to \infty} \int_0^T dT^* \exp\left(-\int_{t_1}^{T^*} dt\, f_c' \right) \int dh \frac{d^N\xi\, d^N\Delta}{(2\pi\sigma^2)^{N+1}} \times \delta_{(\Delta,n)}(\langle \mathbf{x}_1 | \mathbf{x}_0 \rangle) \langle \mathbf{x}_0 | \mathbf{x}_0 \rangle \exp - \frac{[S^* + B(\xi, \Delta)]}{\sigma^2} \tag{B7}$$

where B is the same as in (A12) and $\delta_{(\Delta,n)}$ is

$$\delta_{(\Delta,n)}(\langle \mathbf{x}_1 | \mathbf{x}_0 \rangle) = \delta\left[\langle \dot{z}_c^* | \eta_\Delta \rangle + \sum_{n=1}^{N} \alpha_n \dot{x}_c(t_n^*)\, \xi_n \right] \tag{B8}$$

The important point now is that one can find a set $\{\Delta_0\}$ of parameters so that $dz_c^*/dt = \eta_{\Delta_0}$; thus, the scalar product $\langle dz_c^*/\mathrm{d}t | \eta_\Delta \rangle$ yields a relation between Δ and Δ_0. Moreover, due to the fact that both Q and α_n are arbitrary, one can choose them in order to simplify computations. For instance, for the invariant measure at the unstable fixed point, it suffices to make all the α_n equal to zero, leading to the final formula

$$\mu(0) = \lim_{T \to \infty} \int_0^T dT^* \exp\left(-\int_{t_1}^{T^*} dt\, f_c'\right) \int dh \frac{d^N\xi\, d^N\Delta}{(2\pi\sigma^2)^{N+1}}$$

$$\times \delta(\langle \eta_{\Delta_0} | \eta_\Delta \rangle) \langle \eta_{\Delta_0} | \eta_{\Delta_0} \rangle \exp - \frac{[S^* + B(\xi, \Delta)]}{\sigma^2} \qquad \text{(B9)}$$

In contrast, in calculating the transition probability between two stable fixed points, one cannot choise $\alpha_n = 0$. Indeed, in the part of the trajectory that follows the deterministic system $(u \to s)$, the variable $z(t)$ essentially vanishes, whence the constraint on η degenerates. Therefore, one should retain the constraint on x explicitly (cf. Section 5.3 and Appendix C).

APPENDIX C

Consider the truncated trajectory defined in Fig. 3. To zero order this trajectory is related to the action

$$S^{(0)} = 2S_\infty + \tfrac{1}{2}(z_c, Cz_c)_{\mathbf{I}\times\mathbf{I}} - i[z\,\delta x]^+ \qquad \text{(C1)}$$

where $\mathbf{I} = (-\infty, T_1 + T_2)$ and the last term, associated with the $(+)$ jumps, couples the instanton to the deterministic motion, and is of order $\exp(-b_1 T_1)$ (b_n is the slope of f in the region T_n). To first order we have the linear terms

$$S^{(1)} = (\eta, Cz_c)_{\mathbf{I}\times\mathbf{I}} - (\eta, Cz_c^\infty)_{\mathbf{R}\times\mathbf{R}} - i[\eta\,\delta x]^+ \qquad \text{(C2)}$$

the superscript ∞ means that the limit $T_1 \to \infty$ should be taken. It is worth noticing that the boundary term $[\eta\,\delta x]^+$ contains not only $\exp(-b_1 T_1)$ terms, but also terms of the order unity related to the discontinuity in η at the s_2 stable fixed point. Finally, the second-order action is given by

$$S^{(2)} = \frac{1}{2}(\eta, C\eta)_{\mathbf{I}\times\mathbf{I}} - \sum_{n=1}^{N} \left(i\xi_n \Delta_n + \frac{a_n}{2}\xi_n^2\right) \qquad \text{(C3)}$$

After functional integration we may replace everywhere η by η_Δ. We split the interval $\mathbf{I}$ into $\mathbf{I}^- + \mathbf{I}^+$, where $\mathbf{I}^- = (-\infty, T_0)$ and $\mathbf{I}^+ = (T_0, T_1 + T_2)$. The terms $\mathbf{I}^- \times \mathbf{I}^-$ give the isolated instanton contribution. The crossed terms $\mathbf{I}^- \times \mathbf{I}^+$ are irrelevant in the limit $T \to \infty$, taking into account the ordering $b_1 T_1 \approx 2b_2 T_2$. In the sector $\mathbf{I}^+ \times \mathbf{I}^+$ it is convenient to make a shift $\eta_\Delta^+ \to \eta_\Delta^+ + z_c$, which gives explicitly the contribution of the stable fixed point s_2 to the action [terms in $\exp(2b_2 T_2)$]. Integrating these terms out, one extracts the factor $\mu(s_2)$ of (61). This is possible because the gauge condition can also be split into a $(-)$ sector, where it reduces to that of Section 5.2, and a $(+)$ sector, where, using the fact that z_c essentially vanishes, it only depends on ξ at the discontinuities. The integration over T_2 directly gives a factor T. In turn, the dominant terms that contribute to the other collective degree of motion are in $C\exp(-b_1 T_1)$ as the Jacobian term; then integration over T_1 gives the coefficient C, which cancels with the normalization of the gauge condition. It remains a Gaussian over the reduced bilinear form of (60). In such a way one obtains

$$P(s_2, T \mid s_1) = (T/2\pi)(\det B_R)^{-1/2} \exp(-S_\infty/\sigma^2) \tag{C4}$$

Now, we calculate explicitly the transition time $s_1 \to s_2$ for a white-noise-driven system,

$$f(x) = \begin{cases} -b(x+1) & x \leqslant -b/(a+b) = x_m \\ ax & -x_m < x \leqslant x_m \\ -b(x-1) & x_m < x \end{cases} \tag{C5}$$

The action given by (C1)–(C3) reads

$$\begin{aligned} S = {} & \frac{Z}{2} - Z x_m e^{-aT_1} + \frac{1}{4}\left[\frac{h^2}{b} + \frac{(h+\Delta_1)^2}{a}\right] \\ & + \frac{e^{2bT_2} - 1}{2b}\,[(h+\Delta_1+iZ)\,e^{-aT_1} + \Delta_2]^2 \\ & - i\xi_1\Delta_1 - i\xi_2\Delta_2 - \frac{a_1}{2}\xi_1^2 - \frac{a_2}{2}\xi_2^2 + \delta S \end{aligned} \tag{C6}$$

where

$$\begin{aligned} \delta S = {} & i(h+\Delta_1)\,x_m e^{-aT_1} \\ & - (h+\Delta_1+iZ)^2 \frac{e^{-2aT_1}}{4a} + \frac{iax_m}{b}\,[(h+\Delta_1+iZ)\,e^{-aT_1} + \Delta_2] \end{aligned}$$

The instanton action is $S_\infty = Z/2$, where $Z = 2ab/(a+b)$. The terms δS are negligible taking into account the ordering $1/\sigma^2 \ll aT_1 \approx 2bT_2 \ll \exp(1/\sigma^2)$. The appropriated gauge condition is

$$\delta(\Delta_1)\,\delta(\xi_2)\,Z x_m \tag{C7}$$

Making now the shift $\Delta_2 \to \Delta_2 + (h+\Delta_1+iZ)\exp(-aT_1)$ and integrating over ξ_2, using $\delta(\xi_2)$, we are left with an integration over Δ_2 of the form

$$\int d\Delta_2 \exp\left[-\frac{\exp(2bT_2)}{2b\sigma^2}\Delta_2^2\right] \tag{C8}$$

The Jacobian term is $\exp(-aT_1)\exp(bT_2)$. Using (A25), the integral over (T_2, Δ_2) gives $T/(2\pi\sigma^2 b)^{1/2} = T\mu(s_2)$. In turn, the integration over T_1 reads

$$\int \frac{dT_1}{\sigma^2} \exp(-aT_1) \exp\left[-\frac{Zx_m}{\sigma^2} \exp(-aT_1)\right] \approx \frac{1}{Zx_m} \tag{C9}$$

(higher order terms in σ^2 are neglected) and simplifies with the normalization of (A24). Therefore the transition time is

$$\theta = 2\pi(\det B_R)^{1/2} \exp(S_\infty/\sigma^2) \tag{A31}$$

a result which is formally the same as the one we obtained for arbitrary noise.

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